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The Fermi-Walker derivative and the energy of magnetic striction line with a geodesic Frenet frame

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ABSTRACT

This study investigates the geometric properties of magnetic striction lines on ruled surfaces influenced by magnetic fields, with particular emphasis on the Fermi-Walker derivative and energy of magnetic curves defined in the context of geodesic Frenet frames. That is, the Fermi-Walker derivatives and energy calculations of magnetic curves called magnetic striction lines on ruled surfaces, using the geodesic Frenet frame examined. Theorems are presented that describe when the Fermi-Walker derivatives become zero indicate geometric or physical equilibrium. Analytical formulas for the energy of vector fields are derived based on previously established geometric relations.

KEYWORDS

Fermi-Walker derivatives, Lorentz force, magnetic curves, striction lines.

1. Introduction

The magnetic field is a vector quantity. It is defined by its direction and intensity at any point. Magnetic field has many uses. For example, the earth produces its own magnetic field and forms the basic working principle of the compass. In addition, the rotating magnetic field is used in electric motors and generators. The magnetic field is most generally defined by the Lorentz force acting on a moving electric charge. Lorentz force, in physics, especially in electromagnetism; It is the resultant of electric and magnetic forces on the point charge formed by electromagnetic fields. Some methods have been proposed with the help of Lorentz force. One of them was to monitor the electrical conductivity of biological tissues with electrical impedance tomography. This method has been recently recommended for the diagnosis of early stage cancer tissues.

The magnetic curves on a $(M^n, g)\,,\ (n \geq 2)$ Riemann manifold are the trajectories of charged particles moving on M under the influence of the F magnetic field. F is a closed 2-form magnetic field on M, and the Lorentz force of the magnetic field F on the (M^n, g) manifold is the $(1,1)$-tensor field given by the equation below for all $X, Y \in \chi\,(M^n)$, [Barros and Romero(2007)]

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$$g(\phi(X), Y) = F(X, Y).$$

Approximations of magnetic trajectories corresponding to a Killing vector field associated with a screw movement in 3-dimensional Euclidean space have been studied by Munteanu, [Munteanu(2013)]. Munteanu and Nistor investigated the trajectories of charged particles moving in a homogeneous 3-dimensional Rx2S modeled area under the influence of Killing magnetic fields, [Munteanu and Nistor(2012)]. They explored various methods in the analysis of curves for a given magnetic field at a constant energy level. The Lorentz equation of T-magnetic curves of the V magnetic field was given by Barros et al, [Barros and Romero(2007)]. The solution of this equation has useful applications in many fields such as analytical chemistry, atmospheric science, geochemistry, proton and cancer therapy. Some of the study research in this area can be summarized as follows in 3-dimensional Riemannian manifolds. Barros et al. examined the trajectories obtained by making a variational approach to the magnetic field related to the Killing magnetic field, [Barros et al.(2007)Barros, Cabrerizo, Fernndez and Romero]. Later, the authors suggested that the charge on the particle constantly changes and follows a complex curve with various curvatures, and in their research, they showed that shapes such as flame, hair and tree can be drawn with the help of magnetic curves obtained by this method, [Xu and Mould(2009)]. In 3-dimensional semi-Riemannian manifolds, Özdemir et al. defined the concepts of T-magnetic, N-magnetic and B-magnetic curves and gave some characterizations for these curves, [Özdemir et al.(2015)]. Authors defined a special type of magnetic trajectories associated with a magnetic field defined on a 3D Riemannian manifold, [Körpınar and Demirkol(2018)]. The magnetic flux surfaces and magnetic striction curves produced by with the geodesic Frenet frame in a magnetic field was defined by Bayram et al, [Bayram et al.(2023)]. Recently, the magnetic vector fields and the magnetic flux surfaces along magnetic spherical indicatricies were examined in Minkowski 3-space by Güler et al, [Güler et al.(2023)]. In [Güler(2024)], the author reseach Berry phase model of an optical fiber and the direction of the polarized light with respect to the plane in which the electric field with a geodesic Frenet frame.

The Fermi-Walker derivative has an essential application in geometry and especially in the motions of parallel vector fields. Whether a curve is geodesic or not can be found with the Fermi-Walker derivative. Magnetic curves are a generalization of geodesic curves in a given magnetic field. Therefore, the calculation of Fermi-Walker derivative for magnetic curves plays an important role ([Karakuş and Yaylı(2012)], [Fermi(1922)], [Körpinar et al.(2017)], [Körpinar and Özdemir(2020)]. In addition, by calculating the energy of Fermi-Walker derivatives of magnetic curves, basic definitions and concepts such as mass-energy and motion energy can be better comprehended. To this end; in [Körpinar et al.(2017)], the authors characterized the energy of a charged particle in space. In [Chacón et al.(2001)Chacón, Naveira and Weston], the authors study the energy of the norms of the mean curvatures of an oriented $q-$distribution V and its orthogonal distribution. In this paper, we study the Fermi-Walker derivatives and energy calculations of magnetic curves, called magnetic striction lines on ruled surfaces using the geodesic Frenet frame.

The associated magnetic trajectories are curves $\alpha(s)$ in M^n that satisfy the Lorentz equation

$$\nabla_{\alpha'}\alpha' = \phi\left(\alpha'\right) = V \times \alpha', \tag{1}$$

where ∇ is the Levi-Civita covariant derivative of (M^n, g) and $V(s)$ is a Killing vector field [Barros and Romero(2007)].

The parametric form of a ruled surface is

$$\Psi(s,t) = \alpha(s) + tX(s), \tag{2}$$

where curve $\alpha(s)$ is called the base curve of the ruled surface and $X(s)$ is called the spherical indicatrix or director. The trihedron $\{e_1, e_2, e_3\}$ is called the geodesic Frenet frame of the ruled surface. $e_1 = X \,/\, |X|$, $e_2 = dX/ds$ and $e_3 = e_1 \times e_2$ are unit vector fields and they are called the unit vector field along the director, the center normal vector field, and the asymptotic normal vector field, respectively. The derivative formulas of this frame are

$$\begin{array}{c} e_1' = \frac{1}{R}e_2 \\ e_2' = -\frac{1}{R}e_1 + \frac{\gamma}{R}e_3 \\ e_3' = -\frac{\gamma}{R}e_2 \end{array} \tag{3}$$

where $\gamma = \langle X'' \times X, X' \rangle$ is the geodesic curvature of the spherical indicatrix and $R = |X(s)|$,[Ryuh(1989)].

The tangent vector of the striction line or curve $\beta(s)$ of ruled surface Ψ is

$$\frac{d\beta}{ds} = \Gamma(s)\, e_1(s) + \Delta(s)\, e_3(s), \tag{4}$$

$$\beta(s) = \alpha(s) - \mu(s)\, X(s), \quad \mu = \alpha'(s) \cdot X'(s). \tag{5}$$

and the curvatures of the ruled surface Ψ are, [McCarthy and Roth(1981)].

$$\Gamma(s) = (1/R)\, d\alpha/ds \cdot X - Rd\mu/ds, \;\; \Delta(s) = (1/R)\, d\alpha/ds \cdot X \times de_1/ds \tag{6}$$

Proposition 1. *The Lorentz force according to the geodesic Frenet frame of the e_1 - magnetic striction line of the ruled surface Ψ is*

$$\begin{array}{c} \phi(e_1) = (1/R)\, e_2 \\ \phi(e_2) = -(1/R)\, e_1 + \overline{\Omega}_1 e_3 \\ \phi(e_3) = -\overline{\Omega}_1 e_2, \\ V_1 = \overline{\Omega}_1 e_1 + (1/R)\, e_3, \end{array} \tag{7}$$

where $\overline{\Omega}_1 = g(\phi(e_2), e_3)$, $\overline{\Omega}_1' = 0$, and V_1 is the magnetic Killing vector field along e_1- magnetic striction line, [Bayram et al.(2023)].

The Lorentz force according to the geodesic Frenet frame of the e_2 - magnetic striction line of the ruled surface Ψ is

$$\begin{gathered} \phi(e_1) = (1/R)\, e_2 + \overline{\Omega}_2 e_3, \\ \phi(e_2) = -(1/R)\, e_1 + (\gamma/R)\, e_3, \\ \phi(e_3) = -\overline{\Omega}_2 e_1 - (\gamma/R)\, e_2, \\ V_2 = (\gamma/R)\, e_1 + \overline{\Omega}_2 e_2 + (1/R)\, e_3, \end{gathered} \tag{8}$$

where $\overline{\Omega}_2 = g(\phi(e_1), e_3) = \gamma'$, and V_2 is the magnetic Killing vector field along e_2-magnetic striction line, [Bayram et al.(2023)].

The Lorentz force according to the geodesic Frenet frame of the e_3 - magnetic striction line of the ruled surface Ψ is

$$\begin{gathered} \phi(e_1) = \overline{\Omega}_3 e_2, \\ \phi(e_2) = -\overline{\Omega}_3 e_1 + (\gamma/R)\, e_3, \\ \phi(e_3) = -(\gamma/R)\, e_2, \\ V_3 = (\gamma/R)\, e_1 + \overline{\Omega}_3 e_3, \end{gathered} \tag{9}$$

where $\overline{\Omega}_3 = g(\phi(e_1), e_2)$, $\overline{\Omega}_3' = 0$, V_3 is the magnetic Killing vector field along e_3-magnetic striction line, [Bayram et al.(2023)].

Proposition 2. *Let X be any vector field along the space curve β and T be the tangent vector field of β. If $\nabla_T X = 0$, then the vector field X is said to be parallel in the Fermi walker sense along the space curve β,* [*Benn and Tucker(1989)*].

Definition 1. *Let X be any vector field along the space curve β and T and N be the tangent and normal vectors field of β. The normal Fermi Walker derivative of the vector field X along the space curve β is [Sachs and Wu(1977)],*

$$\widetilde{\nabla}_T X = \nabla_T X - g(N, X)\, \nabla_T N + g(\nabla_T N, X)\, N. \tag{10}$$

2. Fermi Walker derivatives and energies of the magnetic curves with a geodesic Frenet frame

2.1. *Fermi Walker derivatives and energies of the e_1 - magnetic striction line of the ruled surface Ψ*

Proposition 3. *Let β be the e_1 - magnetic striction line of the ruled surface Ψ. The Fermi walker derivatives of the Lorentz forces and Killing vector field V_1 of according to the geodesic Frenet frame of the e_1 - magnetic striction line are*

$$\begin{gathered} \widetilde{\nabla}_{\beta'} \phi(e_1) = \tfrac{1}{R^2} \left[\left(\Gamma^2 - 1 \right) e_1 - \gamma \left(\Delta^2 - 1 \right) e_3 \right], \\ \widetilde{\nabla}_{\beta'} \phi(e_2) = \overline{\Omega}_1 \left(\tfrac{\Delta}{\Gamma} \right)' \Gamma^2 e_1 + \tfrac{1}{R} \left[\left(\Gamma - \Delta \overline{\Omega}_1 \right) (\Gamma - \gamma\Delta) - \left(\tfrac{1}{R} + \gamma \overline{\Omega}_1 \right) \right] e_2 + \tfrac{1}{R} \left(\tfrac{\Delta}{\Gamma} \right)' \Gamma^2 e_3, \\ \widetilde{\nabla}_{\beta'} \phi(e_3) = \tfrac{\overline{\Omega}_1}{R} \left[\left(1 - \Gamma^2 + \Gamma\Delta\gamma \right) e_1 + \left(\gamma\Delta^2 - \gamma - \Delta\Gamma \right) e_3 \right], \\ \widetilde{\nabla}_{\beta'} V_1 = \tfrac{1}{R} \left(\tfrac{\Delta}{\Gamma} \right)' \Gamma^2 e_1 + \left[\tfrac{\overline{\Omega}_1}{R} - \tfrac{\gamma}{R^2} - \left(\tfrac{\Gamma - \gamma\Delta}{R} \right) \left(\tfrac{\Delta}{R} + \Gamma \overline{\Omega}_1 \right) \right] e_2 - \overline{\Omega}_1 \left(\tfrac{\Delta}{\Gamma} \right)' \Gamma^2 e_3. \end{gathered}$$

Proof. If Eqs. (3), (4), (7) is used in Eq. (10) and the necessary derivatives are taken, the desired result is achieved.

Theorem 2.1. *Let β be the e_1 - magnetic striction line of the ruled surface Ψ. The Fermi walker derivatives of the Lorentz forces and Killing vector field of according to the geodesic Frenet frame of the e_1- magnetic striction line are parallel in the asymptotic normal Fermi walker sense along the magnetic striction line .*

□

Proof. Using Eqs. (3), (7), (10), we can write

$$\begin{aligned}\widetilde{\nabla}_{\beta'}^{e_2}\phi(e_1) &= \nabla_{\beta'}\phi(e_1) - g(e_2, \phi(e_1))\nabla_{\beta'}e_2 + g(\nabla_{\beta'}e_2, \phi(e_1))e_2 = 0,\\ \widetilde{\nabla}_{\beta'}^{e_2}\phi(e_2) &= \nabla_{\beta'}\phi(e_2) - g(e_2, \phi(e_2))\nabla_{\beta'}e_2 + g(\nabla_{\beta'}e_2, \phi(e_2))e_2 = 0,\\ \widetilde{\nabla}_{\beta'}^{e_2}\phi(e_3) &= \nabla_{\beta'}\phi(e_3) - g(e_2, \phi(e_3))\nabla_{\beta'}e_2 + g(\nabla_{\beta'}e_2, \phi(e_3))e_2 = 0,\\ \widetilde{\nabla}_{\beta'}^{e_2}V_1 &= \nabla_{\beta'}V_1 - g(e_2, V_1)\nabla_{\beta'}e_2 + g(\nabla_{\beta'}e_2, V_1)e_2 = 0.\end{aligned}$$

□

Theorem 2.2. *Let β be the e_1- magnetic striction line of the ruled surface Ψ. The energy of the Lorentz forces and Killing vector field of according to the geodesic Frenet frame of the e_1- magnetic striction line are*

$$\begin{aligned}\varepsilon(\phi(e_1)) &= \tfrac{1}{2R^4}s + \tfrac{1}{2}\int_\beta\left(\Gamma^2 + \Delta^2 + \tfrac{\gamma^2}{R^4}\right)ds,\\ \varepsilon(\phi(e_2)) &= \tfrac{1}{2R^4}s + \tfrac{1}{2}\int_\beta\left(\Gamma^2 + \Delta^2 + 2\tfrac{\overline{\Omega}_1\gamma}{R^3} + \tfrac{\overline{\Omega}_1^2\gamma^2}{R^2}\right)ds,\\ \varepsilon(\phi(e_3)) &= \tfrac{1}{2}\int_\beta\left(\Gamma^2 + \Delta^2 + \tfrac{\overline{\Omega}_1^2(1+\gamma^2)}{R^2}\right)ds,\\ \varepsilon(V_1) &= \tfrac{1}{2}\int_\beta\left(\Gamma^2 + \Delta^2 + \left(\tfrac{\overline{\Omega}_1}{R} - \tfrac{\gamma}{R^2}\right)^2\right)ds.\end{aligned}$$

Proof. In [Chacón et al.(2001)Chacón, Naveira and Weston], the energy of an oriented q-distribution V in a manifold M is defined. Thus, we can write as follow for the energy of the Lorentz force $\phi(e_1)$ of according to the geodesic Frenet frame of the e_1- magnetic striction line □

$$\begin{aligned}\varepsilon(\phi(e_1)) &= \frac{1}{2}\int_\beta g(\phi(e_1), \phi(e_1))\,ds\\ &= \frac{1}{2}\int_\beta \left(g(\beta', \beta') + g(\nabla_{\beta'}\phi(e_1), \nabla_{\beta'}\phi(e_1))\right)ds,\end{aligned}$$

using Eqs. (4), (7), we have

$$\varepsilon(\phi(e_1)) = \frac{1}{2R^4}s + \frac{1}{2}\int_\beta\left(\Gamma^2 + \Delta^2 + \frac{\gamma^2}{R^4}\right)ds.$$

We write as follow for the energies of the Lorentz forces $\phi(e_2)$ and $\phi(e_3)$ of according to the geodesic Frenet frame of the e_1- magnetic striction line

$$\begin{aligned}
\varepsilon(\phi(e_2)) &= \frac{1}{2}\int_\beta \left(g(\beta',\beta') + g(\nabla_{\beta'}\phi(e_2),\nabla_{\beta'}\phi(e_2))\right) ds \\
\varepsilon(\phi(e_3)) &= \frac{1}{2}\int_\beta \left(g(\beta',\beta') + g(\nabla_{\beta'}\phi(e_3),\nabla_{\beta'}\phi(e_3))\right) ds.
\end{aligned}$$

Using Eqs. (4), (7), desired is achieved.

The energy of the Killing vector field V_1 of according to the geodesic Frenet frame of the e_1- magnetic striction line is

$$\begin{aligned}
\varepsilon(V_1) &= \frac{1}{2}\int_\beta \left(g(\beta',\beta') + g(\nabla_{\beta'}V_1,\nabla_{\beta'}V_1)\right) ds \\
&= \frac{1}{2}\int_\beta \left(\Gamma^2 + \Delta^2 + \left(\frac{\overline{\Omega}_1}{R} - \frac{\gamma}{R^2}\right)^2\right) ds.
\end{aligned}$$

2.2. *Fermi Walker derivatives and energies of the e_2 - magnetic striction line of the ruled surface Ψ*

Proposition 4. *Let β be the e_2 - magnetic striction line of the ruled surface Ψ. The Fermi walker derivatives of the Lorentz forces and Killing vector field V_2 of according to the geodesic Frenet frame of the e_2 - magnetic striction line are*

$$\begin{gathered}
\widetilde{\nabla}_{\beta'}\phi(e_1) = \left[\left(\tfrac{\Delta}{\Gamma}\right)'\Gamma^2\overline{\Omega}_2 + \tfrac{1}{R^2}\left(\Gamma^2 - \gamma\Delta\Gamma - 1\right)\right] e_1 \\
+\tfrac{\overline{\Omega}_2}{R}\left(\gamma\Delta^2 - \Delta\Gamma - \gamma\right) e_2 + \tfrac{1}{R^2}\left(\Delta\Gamma + \gamma\left(1-\Delta^2\right) + \overline{\Omega}_2' R^2\right) e_3 \\
\widetilde{\nabla}_{\beta'}\phi(e_2) = \tfrac{\gamma}{R}\left(\tfrac{\Delta}{\Gamma}\right)'\Gamma^2 e_1 + \tfrac{1}{R^2}\left(\Gamma^2 - \Delta^2\gamma^2 - \gamma^2 - 1\right) e_2 + \left(\tfrac{1}{R}\left(\left(\tfrac{\Delta}{\Gamma}\right)'\Gamma^2 + \gamma'\right)\right) e_3 \\
\widetilde{\nabla}_{\beta'}\phi(e_3) = \left(\tfrac{\gamma}{R^2} - \tfrac{\gamma\Gamma^2}{R} + \tfrac{\gamma^2\Gamma\Delta}{R^2} - \overline{\Omega}_2' - 2\Gamma\Gamma'\overline{\Omega}_2\right) e_1 \\
+\tfrac{1}{R}\left(\Delta\Gamma\gamma\overline{\Omega}_2 - \Gamma^2\overline{\Omega}_2 - 2\overline{\Omega}_2\right) e_2 \\
+\left(\tfrac{\gamma^2}{R^2}\left(\Delta^2 - 1\right) - \tfrac{\Delta\Gamma\gamma}{R} - \overline{\Omega}_2\left(\Gamma\Delta\right)'\right) e_3 \\
\widetilde{\nabla}_{\beta'}V_2 = \tfrac{1}{R}\left(\gamma' - \overline{\Omega}_2 + \overline{\Omega}_2\Gamma^2 - \gamma\Delta\Gamma\overline{\Omega}_2 + \Gamma\Delta'\right) e_1 \\
+\left(\overline{\Omega}_2' + \tfrac{\Gamma\gamma}{R^2}\left(\gamma\Delta - \Gamma\right)\right) e_2 + \left[\tfrac{\gamma}{R}\left(\overline{\Omega}_2\left(1-\Delta^2\right) - \left(\tfrac{\Delta}{\Gamma}\right)'\right) + \tfrac{\Delta}{R}\left(\Delta' + \overline{\Omega}_2\Gamma\right)\right] e_3
\end{gathered}$$

Proof. If Eqs. (3), (4), (8) is used in Eq. (10) and the necessary derivatives are taken, the desired result is achieved. □

Theorem 2.3. *Let β be the e_2 - magnetic striction line of the ruled surface Ψ. If the geodesic curvature γ of the spherical indicatrix is constant, then the Fermi walker derivatives of the Lorentz forces and Killing vector field of according to the geodesic Frenet frame of the e_2- magnetic striction line are parallel in the asymptotic normal Fermi walker sense along the magnetic striction line.*

Proof. Using Eqs. (3), (8), (10), we can write

$$\begin{aligned}
\widetilde{\nabla}^{e_2}_{\beta'}\phi(e_1) &= \nabla_{\beta'}\phi(e_1) - g(e_2,\phi(e_1))\nabla_{\beta'}e_2 + g(\nabla_{\beta'}e_2,\phi(e_1))e_2 = \overline{\Omega}_2'e_3,\\
\widetilde{\nabla}^{e_2}_{\beta'}\phi(e_2) &= \nabla_{\beta'}\phi(e_2) - g(e_2,\phi(e_2))\nabla_{\beta'}e_2 + g(\nabla_{\beta'}e_2,\phi(e_2))e_2 = 0,\\
\widetilde{\nabla}^{e_2}_{\beta'}\phi(e_3) &= \nabla_{\beta'}\phi(e_3) - g(e_2,\phi(e_3))\nabla_{\beta'}e_2 + g(\nabla_{\beta'}e_2,\phi(e_3))e_2 = -\overline{\Omega}_2'e_1 - \tfrac{\overline{\Omega}_2}{R}e_2,\\
\widetilde{\nabla}^{e_2}_{\beta'}V_2 &= \nabla_{\beta'}V_1 - g(e_2,V_1)\nabla_{\beta'}e_2 + g(\nabla_{\beta'}e_2,V_1)e_2 = \tfrac{\overline{\Omega}_2}{R}e_1 + \overline{\Omega}_2'e_2.
\end{aligned}$$

Since the e_2 - magnetic striction line for $\overline{\Omega}_2 = g(\phi(e_1), e_3) = \gamma'$, if the geodesic curvature γ is taken constant, then the Fermi walker derivatives of the Lorentz forces and Killing vector field are zero. Desired is achieved. □

Theorem 2.4. *Let β be the e_2 - magnetic striction line of the ruled surface Ψ. The energy of the Lorentz forces and Killing vector field of according to the geodesic Frenet frame of the e_2- magnetic striction line are*

$$\begin{aligned}
\varepsilon(\phi(e_1)) &= \tfrac{1}{2R^4}s + \tfrac{1}{2}\int_\beta\left(\Gamma^2 + \Delta^2 + \tfrac{\overline{\Omega}_2^2\gamma^2}{R^2} + \left(\tfrac{\gamma}{R^2} + \overline{\Omega}_2'\right)^2\right)ds,\\
\varepsilon(\phi(e_2)) &= \tfrac{1}{2}\int_\beta\left(\Gamma^2 + \Delta^2 + \left(\tfrac{1+\gamma^2}{R^2}\right)^2 + \tfrac{\gamma'^2}{R^2}\right)ds,\\
\varepsilon(\phi(e_3)) &= \tfrac{1}{2}\int_\beta\left(\Gamma^2 + \Delta^2 + \left(\tfrac{\gamma}{R^2} + \overline{\Omega}_2'\right)^2 + \left(\tfrac{\gamma'+\overline{\Omega}_2}{R}\right)^2 + \tfrac{\gamma^2}{R^2}\right)ds,\\
\varepsilon(V_2) &= \tfrac{1}{2}\int_\beta\left(\Gamma^2 + \Delta^2 + \left(\tfrac{\gamma'-\overline{\Omega}_2}{R}\right)^2 + \tfrac{\gamma^2\overline{\Omega}_2^2}{R^2} + \overline{\Omega}_2'^2\right)ds.
\end{aligned}$$

Proof. A similar proof is made to theorem 2.2. □

2.3. *Fermi Walker derivatives and energies of the e_3 - magnetic striction line of the ruled surface Ψ*

Theorem 2.5. *Let β be the e_3 - magnetic striction line of the ruled surface Ψ. The Fermi walker derivatives and energies of the Lorentz forces and Killing vector field V_3 of according to the geodesic Frenet frame of the e_3 - magnetic striction line are follow*

$$\begin{aligned}
\widetilde{\nabla}_{\beta'}\phi(e_1) &= \left(\Gamma\overline{\Omega}_3\left(\tfrac{\Gamma-\gamma\Delta}{R}\right) - \tfrac{\overline{\Omega}_3}{R}\right)e_1 + \left(\Delta\overline{\Omega}_3\left(\tfrac{\Gamma-\gamma\Delta}{R}\right) + \tfrac{\gamma\overline{\Omega}_3}{R}\right)e_2,\\
\widetilde{\nabla}_{\beta'}\phi(e_2) &= \tfrac{\gamma}{R}\left(\tfrac{\Delta}{\Gamma}\right)'\Gamma^2e_1 + \left[\left(\Gamma\overline{\Omega}_3 - \tfrac{\gamma\Delta}{R}\right)\left(\tfrac{\Gamma-\gamma\Delta}{R}\right) - \tfrac{\overline{\Omega}_3}{R} - \tfrac{\gamma^2}{R^2}\right]e_2,\\
\widetilde{\nabla}_{\beta'}\phi(e_3) &= \left(\tfrac{\gamma}{R^2} - \tfrac{\gamma\Gamma}{R}\left(\tfrac{\Gamma-\gamma\Delta}{R}\right)\right)e_1 - \tfrac{\gamma'}{R}e_2 - \left(\tfrac{\gamma^2}{R^2} + \tfrac{\gamma\Delta}{R}\left(\tfrac{\Gamma-\gamma\Delta}{R}\right)\right)e_3,\\
\widetilde{\nabla}_{\beta'}V_3 &= .\left(\tfrac{\gamma'}{R} - \Delta\Gamma'\overline{\Omega}_3\right)e_1 + \left[\tfrac{\gamma}{R^2} - \tfrac{\gamma\overline{\Omega}_3}{R} - \left(\Delta\overline{\Omega}_3 + \tfrac{\gamma\Gamma}{R}\right)\left(\tfrac{\Gamma-\gamma\Delta}{R}\right)\right]e_2 - \tfrac{\gamma}{R}\left(\tfrac{\Delta}{\Gamma}\right)'\Gamma^2e_3.
\end{aligned}$$

and

$$\begin{aligned}
\varepsilon(\phi(e_1)) &= \tfrac{1}{2R^2}s + \tfrac{1}{2}\int_\beta \left(\Gamma^2 + \Delta^2 + \tfrac{\gamma^2}{R^2}\right) ds,\\
\varepsilon(\phi(e_2)) &= \tfrac{1}{2}\int_\beta \left(\Gamma^2 + \Delta^2 + \left(\overline{\Omega}_3 + \tfrac{\gamma^2}{R^2}\right)^2\right) ds,\\
\varepsilon(\phi(e_3)) &= \tfrac{1}{2}\int_\beta \left(\Gamma^2 + \Delta^2 + \tfrac{\gamma^2}{R^4} + \tfrac{\gamma'^2}{R^2} + \tfrac{\gamma^4}{R^4}\right) ds,\\
\varepsilon(V_3) &= \tfrac{1}{2}\int_\beta \left(\Gamma^2 + \Delta^2 + \tfrac{\gamma'^2}{R^2} + \left(\tfrac{\gamma}{R^2} - \tfrac{\gamma\overline{\Omega}_3}{R}\right)^2\right) ds.
\end{aligned}$$

Theorem 2.6. *Let β be the e_3 - magnetic striction line of the ruled surface Ψ. If the geodesic curvature γ of the spherical indicatrix is constant, then the Fermi walker derivatives of the Lorentz forces and Killing vector field of according to the geodesic Frenet frame of the e_3- magnetic striction line are parallel in the asymptotic normal Fermi walker sense along the magnetic striction line.*

Proof. Using Eqs. (3), (9), (10), we can write

$$\begin{aligned}
\widetilde{\nabla}^{e_2}_{\beta'}\phi(e_1) &= \nabla_{\beta'}\phi(e_1) - g(e_2, \phi(e_1))\nabla_{\beta'}e_2 + g(\nabla_{\beta'}e_2, \phi(e_1))e_2 = 0,\\
\widetilde{\nabla}^{e_2}_{\beta'}\phi(e_2) &= \nabla_{\beta'}\phi(e_2) - g(e_2, \phi(e_2))\nabla_{\beta'}e_2 + g(\nabla_{\beta'}e_2, \phi(e_2))e_2 = \tfrac{\gamma'}{R}e_3,\\
\widetilde{\nabla}^{e_2}_{\beta'}\phi(e_3) &= \nabla_{\beta'}\phi(e_3) - g(e_2, \phi(e_3))\nabla_{\beta'}e_2 + g(\nabla_{\beta'}e_2, \phi(e_3))e_2 = -\tfrac{\gamma'}{R}e_2,\\
\widetilde{\nabla}^{e_2}_{\beta'}V_3 &= \nabla_{\beta'}V_3 - g(e_2, V_3)\nabla_{\beta'}e_2 + g(\nabla_{\beta'}e_2, V_3)e_2 = \tfrac{\gamma'}{R}e_1.
\end{aligned}$$

If we take $\gamma = constant$, then the Fermi walker derivatives of the Lorentz forces and Killing vector field are zero. Desired is achieved. □

3. Concluding remarks

Fermi-Walker derivatives and the energy of magnetic curves are important, particularly in studying charged particle motion under magnetic fields. Whether a curve is geodesic or not can be found with the Fermi-Walker derivative. In addition, by calculating the energy of Fermi-Walker derivatives of magnetic curves, basic definitions and concepts such as mass-energy and motion energy can be better comprehended. The cases where the Fermi–Walker derivatives are zero describe the moments when the magnetic striction lines remain "fixed" within the geodesic Frenet frame without rotation and are, therefore, in equilibrium geometrically and physically. Theorems have shown that these equilibrium conditions are directly related to the curvatures of the ruled surface.

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